The real role of displacement current — a history of twists and turns

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A3 Metamaterials Forum 2024, Shenzhen
2024.7.23–25

Purpose of this talk

- The controversy over the meaning of displacement currents has been going on for more than hundred years.
- Separating magnetic fields into those caused by "currents" and those caused by "displacement currents" is not possible in principle.
- The question "does the displacement current create a magnetic field or not?" makes no sense.
- Biot-Savart's equation implicitly includes the effect of displacement current.
- I would like to clarify the confusion surrounding displacement currents and provide an opportunity to end the long standing controversy.

M.K., "Why the controversy over cisplacement currents never ends? "
IEICE Trans. Electron., **E107–C**, 82 (2024)

Displacement current

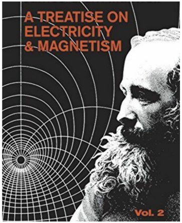
- The time derivative of electric flux density: $\dfrac{\partial \boldsymbol{D}}{\partial t}$
- The final piece to complete the electromagntic theory
  — Maxwell's brilliant insight (only from theory)
- He convinced that light is an electrical and magnetic disturbance.
  — The theoretical progagation speed $(\mu_0 \varepsilon_0)^{-1}$ coincides with the measured value of speed of light.
- Hertz discovered "radio waves" in the process of detecting displacement current using a capacitor.
- Modern textbooks only touch on the topic briefly just before their goal, i.e., Maxwell's equations.

A history

- Ørsted discovered that currents create magnetic fields (1820).
  by observing a compass needle was deflected from magnetic north by a nearby electric current.
- Ampère's circuital law (1820)
  corresponding to Gauss' law
- Biot–Savart law (1820)
  corresponding to Coulomb's law
- Displacement current — Maxwell (1865)
- Light was identified as an electromagnetic disturbance — Maxwell (1865)
- Hertz discovered radio waves (1888)

Treatise

J.C. Maxwell: A Treatise on Electricity and Magnetism
Vol. 2 (Dover, 1954; Clarendon Press, 1891)



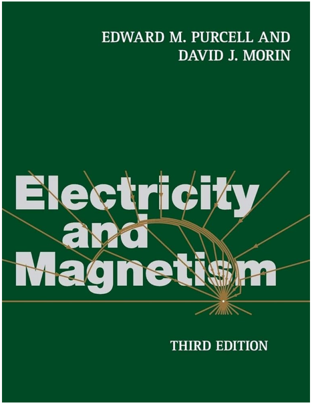
> *One of the chief pecurialities of this treatise is the doctrine which it asserts, that the true electric current* $\mathfrak{C}$ *(*J_{tot}*), that on which the electromagnetic phenomena depend, is not the same thing as* $\mathfrak{K}$ *(*$\boldsymbol{J}$*), the current of conduction, but that the time variation of* $\mathfrak{D}$ *(*$\boldsymbol{D}$*), the electric displacement, must be taken into account in estimating the total movement of electricity, so that we must write,*
>
> $$\mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}}, \qquad \textit{(Equation of True Currents.)}$$

Controversy over displacement current

- While Maxwell's electromagnetic equations have been organized and popularized, a belief that displacement currents do not create magnetic fields emerged.
  The Planck's textbook (1922) may have been the catalyst for this.
- The basis of this belief is "Even if there is a displacement current, the magnetic field can be calculated only from the real current.
- If it is true, what is the displacement current for, and what about electromagnetic waves?
- Various arguments for and against this paradoxical situation have developed.
- There are some textbooks that devote many pages to "Displacement currents do not create magnetic fields."

Purcell's book

[for a capacitor being charged,] the magnetic field $\boldsymbol{B}$, everywhere around the apparatus, is just about what you would expect those conduction currents to produce. In fact, it is almost exactly the field you would calculate if, ignoring the fact that the circuits may not be continuous, you use the Biot-Savart formula [...]

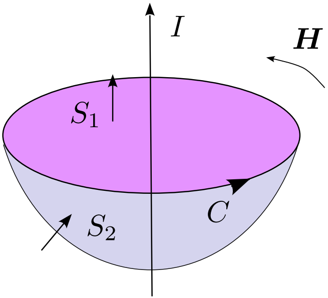


Ampere's law

- Ampère's Law (integral form)

$$\int_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_{S_i} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = I$$

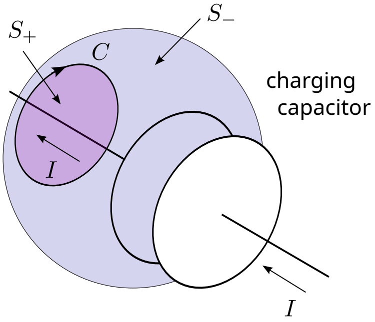
  Closed path C is the edge of surface S_i.



- The surface S_i is arbitrary as long as the closed path C is its edge. $(\partial S_i = C)$
- In a system of axisymmetric currents, the magnetic field is determined with cylindrical coordinates (ρ, ϕ, z).

$$2\pi\rho H_\rho = I, \quad (H_\phi = H_z = 0)$$

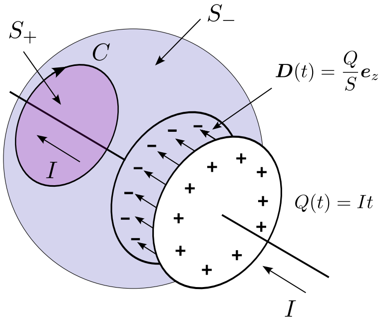
Need for displacement current



- Capacitor charging at constant current I.
- If we try to determine the magnetic field by Ampere's law $\int_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = I$, then

$$I = \int_{S_+} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} \neq \int_{S_-} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = 0$$ Ampère's law does not hold!

Need for displacement current (2)



- Considering the displacement current, we have

$$\int_{S_-} \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot \mathrm{d}\boldsymbol{S} = \frac{\mathrm{d}}{\mathrm{d}t} Q(t) = I$$

Maxwell concluded that what is important in relation to the magnetic field is the total current

$$\boldsymbol{J}_{\mathrm{tot}} := \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Maxwell Ampère's formula

- Ampère-Maxwell formula

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J} + \partial_t \boldsymbol{D}$$

- integral type

$$\oint_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{J} + \partial_t \boldsymbol{D}) \cdot \mathrm{d}\boldsymbol{S}$$

- For a closed curve C, S can be arbitrary as long as $\partial S = C$. In order for the integral to have the same value regardless of the surface S, it is necessary that in all places

$$\boxed{\operatorname{div}(\boldsymbol{J} + \partial_t \boldsymbol{D}) = 0}$$

If the non-steady current $\operatorname{div} \boldsymbol{J} \neq 0$, the a displacement current is required to counteract it.

"Whether displacement currents create a magnetic field or not" is an ill-posed question.

- Neither "the magnetic field created by displacement current" nor "the magnetic field created by current" can be defined. — indivisibility
- If the magnetic field can be split as $\boldsymbol{H} = \boldsymbol{H}_{\boldsymbol{J}} + \boldsymbol{H}_{\partial_t \boldsymbol{D}}$, what is the expression that each of them must satisfy?

$$\operatorname{curl} \boldsymbol{H}_{\boldsymbol{J}} \stackrel{?}{=} \boldsymbol{J}, \quad \operatorname{curl} \boldsymbol{H}_{\partial_t \boldsymbol{D}} \stackrel{?}{=} \partial_t \boldsymbol{D} \qquad (*)$$

- However, the divergence of each of the divided current is

$$0 \stackrel{?}{=} \operatorname{div} \boldsymbol{J}, \quad 0 \stackrel{?}{=} \operatorname{div} \partial_t \boldsymbol{D}$$

  Contradiction!
- Why we introduced the displacement current?

Correct division

- If we want to solve Maxwell-Ampère formula

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J} + \partial_t \boldsymbol{D}$$

by superposition

$$\operatorname{curl} \boldsymbol{H}_1 = \boldsymbol{J}_1 + \partial_t \boldsymbol{D}_1$$
$$\operatorname{curl} \boldsymbol{H}_2 = \boldsymbol{J}_2 + \partial_t \boldsymbol{D}_2$$

the conditions

$$\operatorname{div}(\boldsymbol{J}_1 + \partial_t \boldsymbol{D}_1) = 0$$
$$\operatorname{div}(\boldsymbol{J}_1 + \partial_t \boldsymbol{D}_1) = 0$$

must hold.

The correct way to cut a cake

Notation — Coulomb and dipole fields

- Coulomb field due to a point charge at the origin:

$$\boldsymbol{G}(\boldsymbol{r}) := \frac{\boldsymbol{r}}{4\pi|\boldsymbol{r}|^3}$$

  Derivatives

$$\boldsymbol{\nabla} \cdot \boldsymbol{G}(\boldsymbol{r}) := -\delta^{(3)}(\boldsymbol{r}), \quad \boldsymbol{\nabla} \times \boldsymbol{G}(\boldsymbol{r}) := 0$$

- Flux density for charge q

$$\boldsymbol{D}_q(\boldsymbol{r}) = q\boldsymbol{G}(\boldsymbol{r})$$

- Flux density of the electric dipole $\boldsymbol{p} = q\boldsymbol{l}$

$$\boldsymbol{D}_{\boldsymbol{p}}(\boldsymbol{r}) = -(\boldsymbol{p} \cdot \boldsymbol{\nabla})\boldsymbol{G}(\boldsymbol{r})$$

Bio-Savart's closed-circuit condition

- The magnetic field at $\boldsymbol{r}$ by the current element $I\Delta\boldsymbol{l}$ at the origin

$$\Delta\boldsymbol{H}(\boldsymbol{r}) = I\Delta\boldsymbol{l} \times \boldsymbol{G}(\boldsymbol{r})$$

- Magnetic field due to the current I along a closed path L:

$$\boldsymbol{H}(\boldsymbol{r}) = \oint_L \boldsymbol{H}(\boldsymbol{r}-\boldsymbol{r}') = \oint_L I\mathrm{d}\boldsymbol{l}' \times \boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}')$$

  $\mathrm{d}\boldsymbol{l}'$ is a line element at position $\boldsymbol{r}'$ on path L.

- Originally, the condition that L is closed, i.e. $\partial L = 0$ was required.

Biot-Savart's equation and displacement current

- Vortex of $\Delta \boldsymbol{H}$

$$
\begin{aligned}
\boldsymbol{\nabla} \times \Delta \boldsymbol{H}(\boldsymbol{r}) &= \boldsymbol{\nabla} \times [(I\Delta \boldsymbol{l}) \times \boldsymbol{G}(\boldsymbol{r})] \\
&= I\Delta \boldsymbol{l}\,[\boldsymbol{\nabla} \cdot \boldsymbol{G}(\boldsymbol{r})] - (I\Delta \boldsymbol{l} \cdot \boldsymbol{\nabla})\boldsymbol{G}(\boldsymbol{r}) \\
&= I\Delta \boldsymbol{l}\,\delta^3(\boldsymbol{r}) + \frac{\partial}{\partial t}\boldsymbol{D}_{It\Delta \boldsymbol{l}}(\boldsymbol{r}) =: \Delta \boldsymbol{J}_{\mathrm{tot}}(\boldsymbol{r})
\end{aligned}
$$

- The first term: the original current element $I\Delta \boldsymbol{l}$.
- The second term: the time derivative of the electric flux density due to the electric dipole $\boldsymbol{p}$.

$$
\boldsymbol{D}_{\boldsymbol{p}(t)}(\boldsymbol{r}) = -(\boldsymbol{p}(t) \cdot \boldsymbol{\nabla})\boldsymbol{G}(\boldsymbol{r}), \quad \boldsymbol{p}(t) = It\Delta \boldsymbol{l}
$$

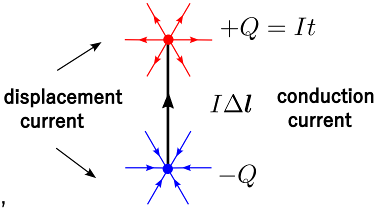
- A current I is applied to the line element $\Delta \boldsymbol{l}$ and charges $\pm Q(t) = \pm It$ are accumulated at both ends $\pm \Delta \boldsymbol{l}/2$.

Bio-Savart's formula (expanded version)

- The total current

$$\Delta \boldsymbol{J}_{\text{tot}} =$$

$$I\Delta \boldsymbol{l}\,\delta^3(\boldsymbol{r}) + \frac{\partial}{\partial t}\boldsymbol{D}_{It\Delta \boldsymbol{l}}(\boldsymbol{r})$$



satisfies the steady-state condition,

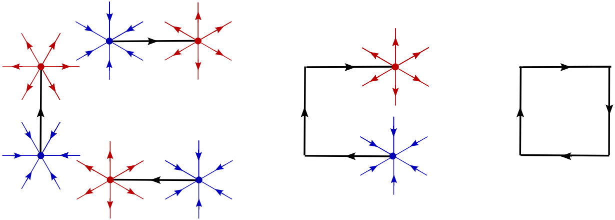
$$\operatorname{div} \boldsymbol{J}_{\text{tot}} = 0$$

- The superposition of the magnetic field elements $\Delta \boldsymbol{H}$ satisfies Maxwell-Ampere's equation

$$\boldsymbol{H}(\boldsymbol{r}) = \int_L \mathrm{d}\boldsymbol{H}(\boldsymbol{r}-\boldsymbol{r}') = \int_L I\mathrm{d}\boldsymbol{l}' \times \boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}')$$

The original constraint that "L is closed" is excessive.

integral of current element



- Integrate for path L whose start (end) point is $\boldsymbol{r}_1$ ($\boldsymbol{r}_2$), we obtain

$$\boldsymbol{J}_{\text{tot}}(\boldsymbol{r}) = \int_L \mathrm{d}\boldsymbol{J}_{\text{tot}}(\boldsymbol{r}-\boldsymbol{r}') = I\int_L \mathrm{d}\boldsymbol{l}'\delta^3(\boldsymbol{r}-\boldsymbol{r}') + I\boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}')\Big|_{\boldsymbol{r}'=\boldsymbol{r}_1}^{\boldsymbol{r}_2}$$

- When current elements are connected, the displacement currents from opposite endpoints cancel each other.
- For a closed path, all displacement currents cancel each other.

Bio-Savart tricks

- Biot-Savart's formula can be modified as $\boxed{\oint \to \int}$, beyond the original assumption, It can be used for unclosed currents.
- Although it is an integral only over the current distribution $\boldsymbol{J}$, the resultant magnetic field is generated by the total current. The total current $\boldsymbol{J}_{\text{tot}} = \boldsymbol{J} + \boldsymbol{J}_{\text{disp}}$ is automatically calculated.
- This "hidden trick" is a cause of confusion over the displacement current.
- The displacement current was introduced by Biot and Savart unconsciously, 45 years before Maxwell's discovery.

Helmholtz decomposition of vector field

- A general three-dimensional vector field can be resolved into the sum of a curl-free (longitudinal) vector field and a divergence-free (transversal) vector field.

$$\boldsymbol{J}(\boldsymbol{r}) = \boldsymbol{J}_{\mathsf{L}}(\boldsymbol{r}) + \boldsymbol{J}_{\mathsf{T}}(\boldsymbol{r}), \quad \mathrm{curl}\, \boldsymbol{J}_{\mathsf{L}}(\boldsymbol{r}) = 0, \quad \mathrm{div}\, \boldsymbol{J}_{\mathsf{T}}(\boldsymbol{r}) = 0$$

- Fourier transformed

$$\tilde{\boldsymbol{J}}(\boldsymbol{k}) = \tilde{\boldsymbol{J}}_{\mathsf{L}}(\boldsymbol{k}) + \tilde{\boldsymbol{J}}_{\mathsf{T}}(\boldsymbol{k}), \quad \boldsymbol{k} \times \tilde{\boldsymbol{J}}_{\mathsf{L}}(\boldsymbol{k}) = 0, \quad \boldsymbol{k} \cdot \tilde{\boldsymbol{J}}_{\mathsf{T}}(\boldsymbol{k}) = 0$$

The current given by Biot-Savart's magnetic field

- Biot-Savart equation for 3D current density $\boldsymbol{J}(\boldsymbol{r})$

$$\boldsymbol{H}(\boldsymbol{r}) = \int_V \mathrm{d}v' \boldsymbol{J}(\boldsymbol{r}') \times \boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}')$$

The vortex of this magnetic field is

$$\mathrm{curl}\,\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{J}(\boldsymbol{r}) - \boxed{\int_V \mathrm{d}v' (\boldsymbol{\nabla}' \cdot \boldsymbol{J}(\boldsymbol{r}')) \boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}')}$$

- From the charge conservation law $\boldsymbol{\nabla} \cdot \boldsymbol{J}(\boldsymbol{r}) + \partial_t \varrho(\boldsymbol{r}, t) = 0$, the 2nd term is identified as displacement current;

$$\int_V \mathrm{d}v' \partial_t \varrho(\boldsymbol{r}', t) \boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}') = \partial_t \boldsymbol{D}(\boldsymbol{r}, t)$$

Projection operators

- The 2nd term is also identified as the longitudinal component of the current density $\boldsymbol{J}(\boldsymbol{r})$.

$$\boldsymbol{J}_{\mathsf{L}} = (\hat{L}\boldsymbol{J})(\boldsymbol{r}) := \int \mathrm{d}v' (\boldsymbol{\nabla}' \cdot \boldsymbol{J}(\boldsymbol{r}')) \boldsymbol{G}(\boldsymbol{r} - \boldsymbol{r}')$$

- The opertors $\hat{L}$ and $\hat{T} := \hat{1} - \hat{L}.$ are projection operators;

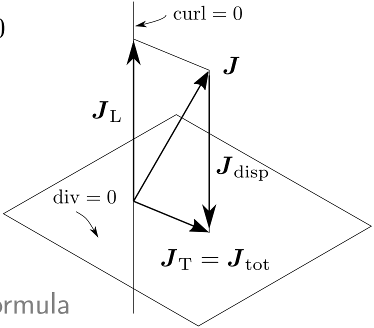
$$\hat{T}^2 = \hat{T}, \quad \hat{L}^2 = \hat{L}, \quad \hat{T}\hat{L} = \hat{L}\hat{T} = 0$$

- The vortex of Biot-Savart field can be expressed as

$$\mathrm{curl}\, \boldsymbol{H} = \boldsymbol{J}(\boldsymbol{r}) - \hat{L}\boldsymbol{J}(\boldsymbol{r}) = \hat{T}\boldsymbol{J}(\boldsymbol{r})$$

- Inverting the relation, we have

$$\boldsymbol{H} = (\mathrm{curl}^{-1})(\hat{T}\boldsymbol{J}) \quad \text{Biot-Savart formula}$$



Functions of Biot-Savart's formula

- Biot-Savart's equation is a function of the transverse component of the current density

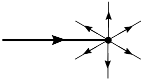
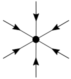
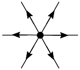
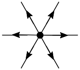
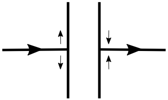
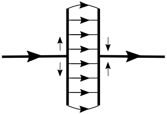
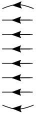
$$\boldsymbol{J}_{\mathsf{T}} = \boldsymbol{J} - \boldsymbol{J}_{\mathsf{L}}$$

- That is, the magnetic field due to the total current density

$$\boldsymbol{J}_{\text{tot}} := \boldsymbol{J} + \boldsymbol{J}_{\text{disp}}$$

  is calculated.
- The difference between the difference and the sum gives a very different impression.

Helmholtz decomposition of for open current distributions

	$\boldsymbol{J}$	$=$	$\boldsymbol{J}_{\mathsf{T}}$	$+$	$\boldsymbol{J}_{\mathsf{L}}$
	$\boldsymbol{J}$	$=$	$\boldsymbol{J}_{\mathsf{tot}}$	$+$	$(-\boldsymbol{J}_{\mathsf{disp}})$
Infinitesimal	$\boldsymbol{p}\,\delta^3(\boldsymbol{r})$		$(\boldsymbol{p} \times \boldsymbol{\nabla}) \times \boldsymbol{G}(\boldsymbol{r})$		$(\boldsymbol{p} \cdot \boldsymbol{\nabla})\boldsymbol{G}(\boldsymbol{r})$
Semi-infinite linear current					
Spherically symmetric current			0		
Capacitor					

Electromagnetic waves and displacement currents

- The magnetic field is transverse: $\operatorname{div} \boldsymbol{B} \equiv 0 \Rightarrow \boldsymbol{B}_{\mathsf{L}} = 0$.
- The electric field has both vertical and horizontal components. The vertical component is $\operatorname{div} \boldsymbol{D}_{\mathsf{L}} = \varrho$.
- The Helmholtz decomposition of Maxwell's equations.

$$\operatorname{curl} \boldsymbol{E}_{\mathsf{T}} = -\partial_t \boldsymbol{B}_{\mathsf{T}}$$
$$\operatorname{curl} \boldsymbol{H}_{\mathsf{T}} = \operatorname{curl} \boldsymbol{J}_{\mathsf{T}} + \partial_t \boldsymbol{D}_{\mathsf{T}} + \cancel{\boldsymbol{J}_{\mathsf{L}} + \partial_t \boldsymbol{D}_{\mathsf{L}}}$$

  From the charge conservation law $\operatorname{div}(\boldsymbol{J} + \partial_t \boldsymbol{D}) = 0$ we have $\boldsymbol{J}_{\mathsf{L}} + \partial_t \boldsymbol{D}_{\mathsf{L}} = 0$.
- Since either $\boldsymbol{J}_{\mathsf{T}}$ and $\partial_t \boldsymbol{D}_{\mathsf{T}}$ are transverse we can say that each of them creates a magnetic field (correct partitioning).
- We can safely say the transverse component of the displacement current $\partial_t \boldsymbol{D}_{\mathsf{T}}$ creates a magnetic field.

Electromagnetic wave

- Summary

$$\operatorname{curl} \boldsymbol{E}_{\mathsf{T}} = -\partial_t \boldsymbol{B}_{\mathsf{T}}$$
$$\operatorname{curl} \boldsymbol{H}_{\mathsf{T}} = \partial_t \boldsymbol{D}_{\mathsf{T}} + \boldsymbol{J}_{\mathsf{T}}$$

- The transverse electric field ($\boldsymbol{D}_{\mathsf{T}} = \varepsilon_0 \boldsymbol{E}_{\mathsf{T}}$ causes entanglement of the transverse electric and magnetic fields, which allows electro-magnetic modes.
- In particular, even when $\boldsymbol{J}_{\mathsf{T}} = 0$, a free solution is possible.
- Because $\partial_t \boldsymbol{D}_{\mathsf{T}}$ is not bound by the real current, the wave can propagate far from the source.

Summary

- The long-standing controversy over the significance of displacement currents has been mired in the dichotomy of to create a magnetic field or not to create a magnetic field.
- In the case of non-steady current, neither "magnetic field created by current" nor "magnetic field created by displacement current" can be defined.
- The effect of displacement current is automatically incorporated in the magnetic field calculated by Biot-Savart's formula.
- The role of the longitudinal component and the transverse component is different.